11 Let $x, y \in \mathbb{R}$. Show that $x^2 + y^2 \geq 2xy$.

Assuming $x^2 + y^2 \geq 2xy \Rightarrow x^2 + y^2 - 2xy \geq 0$

We have $x^2 + y^2 - 2xy$

$$= x^2 - 2xy + y^2$$

$$= \underbrace{(x - y)^2}_{\text{always positive}}$$

≥ 0 as required

$\therefore x^2 + y^2 \geq 2xy$

14 **a** Consider the numbers $\frac{9}{10}$ and $\frac{10}{11}$. Which is larger?

b Let n be a natural number. Prove that $\frac{n}{n+1} > \frac{n-1}{n}$.

Assuming $\frac{n}{n+1} > \frac{n-1}{n} \Rightarrow \frac{n}{n+1} - \frac{n-1}{n} > 0$

We have $\frac{n}{n+1} - \frac{n-1}{n}$

$$= \frac{n}{(n+1)} \times \frac{n}{n} - \frac{n-1}{n} \times \frac{(n+1)}{(n+1)}$$

$$= \frac{n^2}{n(n+1)} - \frac{(n-1)(n+1)}{n(n+1)}$$

$$= \frac{n^2}{n(n+1)} - \frac{n^2-1}{n(n+1)}$$

$$= \frac{n^2-(n^2-1)}{n(n+1)}$$

$$= \frac{1}{n(n+1)}$$

> 0 as required since denominator will be a positive number ≥ 1 since n is a natural number

$$\therefore \frac{n}{n+1} > \frac{n-1}{n}$$

16 Let $a, b \in \mathbb{R}$. Prove that $\dfrac{a^2+b^2}{2} \geq \left(\dfrac{a+b}{2}\right)^2$.

Assuming $\dfrac{a^2+b^2}{2} \geq \left(\dfrac{a+b}{2}\right)^2 \Rightarrow \dfrac{a^2+b^2}{2} - \left(\dfrac{a+b}{2}\right)^2 \geq 0$

we have $\dfrac{a^2+b^2}{2} - \left(\dfrac{a+b}{2}\right)^2$

$$= \frac{(a^2+b^2)}{2} \times \frac{2}{2} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{2a^2+2b^2}{4} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{a^2-2ab+b^2}{4}$$

$$= \frac{(a-b)^2}{4}$$

≥ 0 since numerator always positive

$$\therefore \frac{a^2+b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$$